Notes on indifference pricing of derivatives in asymptotically complete market [1]

Dapeng Shang [2]

1. Introduction

In this notes, we study the indifference price for derivatives in an asymptotically complete market. To achieve this goal, a generalized version of Gärtner-Ellis theorem is needed to analyze indifference price behavior in the limit. We consider an optimal investing problem with one risky asset and one contingent claim in the market where risk-aversion investors are characterized by exponential utility. Both trinomial and Gaussian models in term of risk asset return are provided to get an insight into the generalized Gärtner-Ellis theorem.

2. Large deviation principles theory

The main results from large deviation theory, including large deviations principle, Cramér's and Gärtner-Ellis Theorem, are stated in this part. In the next prat, we will try to connect the indifference price with those results.

Let $(\mathscr{S}, \mathscr{B})$ be a measure space and $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ be a family of probability measures.$\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviations principle (LDP) if, for all $\Gamma \in \mathscr{B}$,

$$\exists\, I : \mathscr{S} \to [0,\infty] \text{ s.t. } -\inf_{x\in\Gamma^\circ} I(x) \leq \underline{\lim}_{\varepsilon\to 0}\, \varepsilon \log \mathbb{P}_\varepsilon(\Gamma) \leq \overline{\lim}_{\varepsilon\to 0}\, \varepsilon \log \mathbb{P}_\varepsilon(\Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x) \tag{1}$$

For $\mathscr{B}_{\mathscr{S}} \subset \mathscr{B}$ (all the open sets are measurable), the LDP equavalent to

$$\overline{\lim}_{\varepsilon\to 0}\, \varepsilon \log \mathbb{P}\varepsilon(F) \leq -\inf_{x\in F} I(x) \text{ for all } F \text{ closed} \tag{2}$$

$$\underline{\lim}_{\varepsilon\to 0}\, \varepsilon \log \mathbb{P}\varepsilon(G) \geq -\inf_{x\in G} I(x) \text{ for all } G \text{ open} \tag{3}$$

2.1 Rate function and weak large deviations principle

We define a function I is lower semicontinuous (l.s.c) as $\{I \leq \alpha\}$ is compact for all $\alpha \in \mathbb{R}$ for $I : \mathscr{S} \to [0,\infty]$. Then, let $(\mathscr{S}, \tau)$ be a topological space and $I : \mathscr{S} \to [0,\infty]$ is a rate function if it is lower semicontinuous. Furthermore, I is a good rate function if $\{I \leq \alpha\}$ is compact for all $\alpha \geq 0$.

The weak LDP in the Borel case is a slight relaxation,

$$\overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\varepsilon(F) \leq -\inf_{x \in F} I(x) \text{ for all } F \text{ compact} \tag{4}$$

$$\underline{\lim}\, \varepsilon \to 0 \varepsilon \log \mathbb{P}\varepsilon(G) \geq -\inf_{x \in G} I(x) \text{ for all } G \text{ open} \tag{5}$$

A family $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ is exponentially tight if, for every $0 < \alpha < \infty$, there is a compact set K_α such that $\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_\varepsilon \left(K_\alpha^{\mathrm{c}}\right) < -\alpha$. And, if $\mathscr{S}$ is polish (separable and complete), then the LDP and goodness of I imply exponentially tight.

Lemma : For $\mathscr{B}_{\mathscr{S}} \subset \mathscr{B}$,

1. if an exponentially tight family $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ satisfies a weak LDP then it satisfies a full LDP;
2. if an exponentially tight family $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ satisfies the LDP lower bound for open sets, for a rate function I, then I is a good rate function.

To show a family $\{\mathbb{P}_n\}$ satisfies a full LDP, one can just show it is exponentially tight and satisfies a weak LDP.

2.2 Cramér's Theorem

The goal for Cramér's theorem is giving conditions which make the empirical average of i.i.d. random variables satisfy an LDP.

Let $\{X_i\}_i^\infty$ be $\mathbb{R}^d$ -valued i.i.d. random vectors with distribution μ under $\mathcal{P}$, and let $S_n := \frac{1}{n} \sum_{i=1}^n X_i$ be the empirical average. Let μ_n be the distribution of S_n, a probability measure on $\left(\mathbb{R}^d, \mathscr{B}\left(\mathbb{R}^d\right)\right)$.

The question is whether the family $\{\mu_n\}_{n=1}^\infty$ satisfies an LDP, and if so, what is the rate function?

Let's build the rate function from the cumulant generating function

$$\Lambda(\lambda) := \log \mathbb{E}\left[\exp(\lambda, X_1))\right] \tag{6}$$

where the domain of Λ is $\mathscr{D}_\Lambda := \left\{\lambda \in \mathbb{R}^d : \Lambda(\lambda) < \infty\right\}$.

The Fenchel-Legendre transform of Λ is

$$\Lambda^*(x) := \sup_{\lambda \in \mathbb{R}^d} \{(\lambda, x) - \Lambda(\lambda)\} \tag{7}$$

Define $\mathscr{D}_{\Lambda^*} := \left\{x \in \mathbb{R}^d : \Lambda^*(x) < \infty\right\}$ as the domain of Λ^*. Here, one can prove that $\Lambda^*(x)$ is a rate function. If 0$\in \mathscr{D}^\circ$ then Λ^* is a good rate function. Then, Cramér's theorem are stated as follow,

Theorem : If 0$\in \mathscr{D}^\circ_{\text{then}}$ $\{\mu_n\}_{n=1}^\infty$ satisfies the LDP with good rate function Λ^*

2.3 Gärtner-Ellis theorem

Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of random variables, where for each n, Z_n is distributed as μ_n, with cumulant generating function Λ_n. Assume that, for all $\lambda \in \mathbb{R}^d$, $\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{n}\Lambda_n(\lambda)$ exists as an extended real number, and $0 \in \mathscr{D}_\Lambda^\circ$. Then, Gärtner-Ellis theorem gives

1. $\Lambda^*(x)$ is a good rate function as $0 \in \mathscr{D}_\Lambda^\circ$
2. $\overline{\lim}_{n\to\infty} \frac{1}{n}\log \mu_n(\Gamma) \leq -\inf_{y\in\Gamma} \Lambda^*(y)$, where Γ is compact;
3. $\{\mu_n\}$ is exponentially tight, so 1) holds for any closed set F
4. For any open set G, $\underline{\lim}_{n\to\infty} \frac{1}{n}\log \mu_n(G) \geq -\inf_{y\in G\cap\mathscr{Y}} \Lambda^*(y)$

If we can find $\lambda^* =: \lambda_y$ s.t $((\lambda^*, y) - \Lambda^*(y)) - ((\lambda^*, z_0) - \Lambda^*(z_0)) > 0$ then y is said to be an exposed point of Λ^*, with exposing hyperplane λ_y. Define $\mathscr{Y} := \{y : y$ is an exposed point with an exposing hyperplane λ_y and $\lambda_y \in \mathscr{D}_\Lambda^\circ\}$.

This includes the setting for Cramér's theorem as a special case.

3. Market with Derivative Securities

We consider an investor in the financial market with initial wealth $X_0 = x_0$, and the wealth process X_t^π using portfolio strategy π. His or her position in derivative is τ, and the payoff of derivative at terminal time T is h. We assume interest rate $r = 0$.

The optimal utility problem for this investor is,

$$\mathcal{U}(x_0, \tau) = \sup_\pi \mathbb{E}[U(X_T^\pi + \tau h)] \tag{8}$$
$$\text{s.t. } X_0 = x_0 \tag{9}$$
$$\mathbb{E}[X_T^\pi z_T] \leq x_0, \forall\pi \text{ and for any random variable } z_T > 0 \tag{10}$$

Define $V(y) = \sup_{x\in\mathbb{R}}(U(x) - xy)$, then we can transfer the original optimal problem into its dual problem.

Proposition 1 : The dual problem for optimal problem (8) with constrains,

$$\sup_\pi \mathbb{E}[U(X_t^\pi + \tau h)] \leq \inf_{z_T\in\mathscr{Z}} \inf_{\lambda>0} (\mathbb{E}[V(\lambda z_T) + \lambda(x_0 + \tau\mathbb{E}[hz_T])]) \tag{11}$$

where $\mathscr{Z} = \{z_T :$ postive random variable s.t. $\mathbb{E}[X_t^\pi z_T] \leq x_0, \forall\pi\}$.

The equality holds if and only if (π, λ, z_T) s.t.

$$X_T^\pi = I(\lambda z_T) - \tau h \tag{12}$$

where $I(\cdot)$ is the inverse function of first order derivative of utility function U, and

$$\mathbb{E}[X_T^\pi z_T] = x_0 \tag{13}$$

Proof : By the definition of $V(y)$, we have $U(x) \leq V(y) + xy$. Take $x = X_t^\pi(\omega) + \tau h(\omega), y = \lambda z_t(\omega)$, then,

$$U(X_T^\pi + \tau h) \leq V(\lambda z_T) + \lambda z_T (X_T^\pi + \tau h) \text{ a.s.}$$

where the equality holds if and only if $\dot{U}(X_T^\pi + \tau h) = \lambda z_T$ a.s.. Take expecation from both sides and consider (9) and (10), for any $\lambda > 0$ and $z_T \in \mathscr{Z}$

$$\begin{aligned}\mathbb{E}[U(X_T^\pi + \tau h)] &\leq \mathbb{E}[V(\lambda z_T) + \lambda z_T (X_T^\pi + \tau h)] \\ &\leq \lambda x_0 + \lambda \tau \mathbb{E}[z_T h] + \mathbb{E}[V(\lambda z_T)]\end{aligned}$$

the equality holds if and only if $\mathbb{E}[X_T^\pi z_T] = x_0$. Then, one can take the supreme over π on the right hand side and infimum over z_T and λ, (11) proved. □

3.1 Exponential utility

In this notes, we take a specific form of utility: $U(x) = -e^{-\gamma x}$, where γ measures risk-averse level. Assume initial wealth $x_0 = 0$ and $\mathbb{E}[z_T] = 1$. In this setting, we obtained $V(y) = \frac{y}{\gamma}(log(\frac{y}{\gamma}) - 1)$ and dual problem

$$U(\inf_{z_T \in \mathscr{Z}} \{\tau \mathbb{E}[z_T h] + \frac{1}{\gamma} \mathbb{E}[z_T \log(z_T)]\}) \tag{14}$$

3.2 Indifference price

The indifference price p_I for derivatives in the market is the per-unit buying price which equals the utility for paying $p_I \tau$ to get τ unit of h with doing nothing.

For exponential utility,

$$\mathcal{U}(x_0, \tau) = \sup_\pi \mathbb{E}[-e^{-\gamma(X_T^\pi + \tau h)}] = \sup_\pi \mathbb{E}[-e^{-\gamma(x_0 + \int_0^T \pi_t' dS_t + \tau h)}] = -e^{-\gamma x_0} \mathcal{U}(0, \tau) \tag{15}$$

Then the indifferen price in this case would equals $\mathcal{U}(x_0 - p_I \tau, \tau) = \mathcal{U}(x_0, 0)$, which turns out to be a function of position τ,

$$p_I(\tau) = -\frac{1}{\gamma \tau} \log(\frac{\mathcal{U}(0, \tau)}{\mathcal{U}(0, 0)}) \tag{16}$$

As $\mathcal{U}(0, \tau) = \sup_\pi \mathbb{E}[U(X_T^\pi + \tau h)]$ with $x_0 = 0$, by (14), we can express it in the dual form

$$\mathcal{U}(0, \tau) = -e^{-\gamma(\inf_{z_T \in \mathscr{Z}} \{\tau \mathbb{E}[z_T h] + \frac{1}{\gamma} \mathbb{E}[z_T \log(z_T)]\})} \tag{17}$$

Also,

$$\mathcal{U}(0, 0) = -e^{-\gamma(\inf_{z_T \in \mathscr{Z}} \{\frac{1}{\gamma} \mathbb{E}[z_T \log(z_T)]\})} = -e^{\mathbb{E}[z_T^0 \log(z_T^0)]} \tag{18}$$

Here, z_T^0 optimizes $\frac{1}{\gamma} \mathbb{E}[z_T \log(z_T)]$ over set $\mathscr{Z}$. Therefore, the indifference price is

$$p_I(\tau) = \frac{1}{\tau}(\inf_{z_T \in \mathscr{Z}} \{\tau \mathbb{E}[z_T h] + \frac{1}{\gamma}(\mathbb{E}[z_T \log(z_T)] - \mathbb{E}[z_T^0 \log(z_T^0)])\}) \tag{19}$$

We introduce the "minimal entropy measure" $\mathbb{Q}^0$ via z_T^0,

$$\mathbb{Q}^0 : z_T^0 = \frac{d\mathbb{Q}^0}{d\mathbb{P}} | \mathcal{F}_T \tag{20}$$

p_I can be transformed in term of relative entropy,

$$\begin{aligned} \tau p_I(\tau) &= \inf_{z_T \in \mathscr{Z}} \{\tau \mathbb{E}[z_T h] + \frac{1}{\gamma}(\mathbb{E}[z_T \log(z_T)] - \mathbb{E}[z_T^0 \log(z_T^0)])\} \\ &= \inf_{\mathbb{Q}} \{\tau \mathbb{E}^{\mathbb{Q}}[h] + \frac{1}{\gamma}(H(\mathbb{Q}|\mathbb{P}) - H(\mathbb{Q}^0|\mathbb{P}))\} \\ &\leq \tau \mathbb{E}^{\mathbb{Q}^0}[h] \end{aligned}$$

From above inequality, it is easy to see that $p_I(\tau)$ converges to h when τ fixed and $\mathbb{P} \to 0$. However, the question we want to ask is how does the indifference price behave if there is a large positon in derivatievs, i.e. $\tau \to \infty$. To answer this question, we connect it with large deviation theory and an extension to Gärtner-Ellis theorem is needed.

3.3 Connection between indifference price and large deviation theory

For any $\tau \in \mathbb{R}$, there exists an unique optimal portfolio $\hat{\pi}(\tau)$. Denote $X_T^{\hat{\pi}(\tau)}$ as $\hat{X}_T^{\tau}$, then for $\tau = 0$,

$$\mathcal{U}(0,0) = U(\mathbb{E}[z_T^0 \log(z_T^0)]) = \sup_{\pi} \mathbb{E}[U(X_T^{\pi})] = \mathbb{E}[U(\hat{X}_T^0)]$$

Therefore, z_T^0 and $\hat{X}_T^0$ satisfies : 1) $\dot{U}(\hat{X}_T^0) = \lambda z_T^0$ and 2) $\mathbb{E}[z_T^0] = 1$. Combine these two conditions, z_T^0 can be solved as,

$$z_T^0 = \frac{e^{-\gamma \hat{X}_T^0}}{\mathbb{E}[e^{-\gamma \hat{X}_T^0}]} \tag{21}$$

Now, we can change into minimal entropy measure,

$$\begin{aligned} \frac{\mathcal{U}(0,\tau)}{\mathcal{U}(0,0)} &= \frac{\mathbb{E}[-e^{-\gamma \hat{X}_T^{\tau} - \gamma \tau h}]}{\mathbb{E}[-e^{-\gamma \hat{X}_T^0}]} = \mathbb{E}[e^{-\gamma \tau [\frac{1}{\tau}(\hat{X}_T^{\tau} - \hat{X}_T^0) + h]} \times \frac{e^{-\gamma \hat{X}_T^0}}{\mathbb{E}[e^{-\gamma \hat{X}_T^0}]}] \\ &= \mathbb{E}[e^{-\gamma \tau [\frac{1}{\tau}(\hat{X}_T^{\tau} - \hat{X}_T^0) + h]} \times z_T^0] = \mathbb{E}[e^{-\gamma \tau [\frac{1}{\tau}(\hat{X}_T^{\tau} - \hat{X}_T^0) + h]} \times \frac{d\mathbb{Q}^0}{d\mathbb{P}} | \mathcal{F}_T] \\ &= \mathbb{E}^0[e^{-\gamma \tau Y(\tau)}] \end{aligned}$$

where $Y(\tau) = \frac{1}{\tau}(\hat{X}_T^{\tau} - \hat{X}_T^0) + h$. And, under $\mathbb{Q}^0$, the indifference price is

$$p_I(\tau) = -\frac{1}{\gamma \tau} \log \mathbb{E}^0[e^{-\gamma \tau Y(\tau)}] \tag{22}$$

For large position $\tau \to \infty$, we can take a sequence of $\tau_n = l \cdot r_n$ and define a sequence of indiffrent price

$$p_I^n(lr_n) = -\frac{1}{\gamma l r_n} \log \mathbb{E}_n^0[e^{-\gamma l r_n Y_n(lr_n)}] \tag{23}$$

To explore the limiting behavor, we need to find the right rate r_n such that $p_I^n(lr_n) \to p^\infty(l)$ when $r_n \to \infty$.

3.4 Differences between Laplace theorem , Gärtner-Ellis theorem and our goal

For Laplace theorem, Y is a random variable, when $r_n \to \infty$,

$$\frac{1}{lr_n} \log \mathbb{E}[e^{lr_n Y}] \to \begin{cases} \text{ess sup}\, Y \ , \ \text{for } l > 0 \\ \text{ess inf}\, Y \ , \ \text{for } l < 0 \end{cases} \tag{24}$$

For Gärtner-Ellis theorem, Y_n is sequence of random variable, when $r_n \to \infty$, if

$$\frac{1}{lr_n} \log \mathbb{E}[e^{lr_n Y_n}] \to \Lambda(l) \tag{25}$$

and $0 \in \mathscr{D}_\Lambda^\circ$, then μ_n, the distribution of Y_n, has LDP with rate function $\Lambda^*(l)$, the Fenchel-Legendre transform of Λ.

For our model, $Y_n(lr_n)$ is a function of r_n, when $r_n \to \infty$, we want to identify a generalized Gärtner-Ellis theorem in terms of that

$$\frac{1}{lr_n} \log \mathbb{E}[e^{lr_n Y_n(lr_n)}] \to p^\infty(l) \tag{26}$$

and some condition with respect to $\mathbb{P}[(Y_n|l) \in A]$

4. Trinomial Case

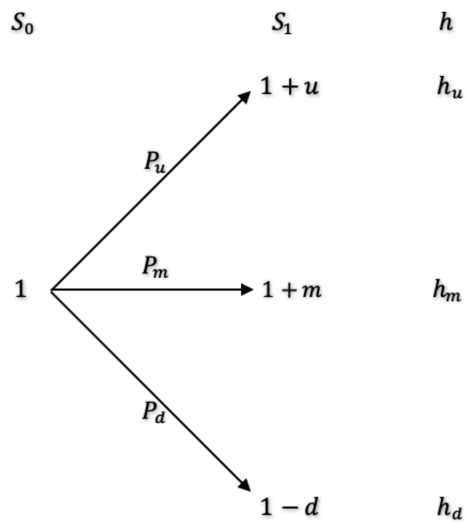

Figure 1. Trinomial Model

To get an insight into our generalized Gärtner-Ellis theorem, we consider a one period trinomial tree model. As figure 1 shows, there are three states $\{\text{up}, \text{middle}, \text{down}\}$, associated with probability $\{P_u, P_m, P_d\}$. The underly asset price is 1 at time 0, and increases $\{u, m, -d\}$ for each state. The payoff of the derivative are $\{h_u, h_m, h_d\}$ in each state. To ensure the no arbitrage conditions, we assume that $u > m > -d$ and $d > 0$.

In this scenario, the original optimal utility problem turns into,

$$\sup_{\pi} -[P_u e^{-\gamma(\pi u+\tau h_u)} + P_m e^{-\gamma(\pi m+\tau h_m)} + P_d e^{-\gamma(-\pi d+\tau h_d)}] \tag{27}$$

Take the first order condition,

$$0 = g(\pi) := uP_u e^{-\gamma(\pi u+\tau h_u)} + mP_m e^{-\gamma(\pi m+\tau h_m)} - dP_d e^{-\gamma(-\pi d+\tau h_d)} \tag{28}$$

As $g'(\pi) < 0$ for sure, $g(\pi)$ is a decreasing function in π with $g(+\infty) = -\infty$ and $g(-\infty) = +\infty$. So, there exists a portfolio strategy $\hat{\pi} = \pi(\tau)$ which set $g(\pi) = 0$.

Proposition 2 : The optimal portfolio strategy is

$$\hat{\pi}(\tau) = \hat{\pi}^0(\tau) + \theta \tag{29}$$

where $\hat{\pi}^0(\tau) = \frac{h_d - h_u}{u+d}\tau + \frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d})$ and θ satisfies $e^{\gamma\theta(m+d)} - e^{-\gamma\theta(u-m)} = \frac{mP_m}{\bar{K}} e^{\gamma\tau\bar{H}}$.

If $m = \frac{u-d}{2}$, then

$$\hat{\pi}(\tau) = \frac{h_d - h_u}{u+d}\tau + \frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d}) + \frac{2}{\gamma(u+d)}\sinh^{-1}(\frac{mP_m}{2\bar{K}} e^{\gamma\tau\bar{H}}) \tag{30}$$

by defining $\bar{K} = (uP_u)^{\frac{m+d}{u+d}} (dP_d)^{-\frac{m-u}{u+d}}$ and $\bar{H} = \frac{m+d}{u+d} h_u + \frac{u-m}{u+d} h_d - h_m$.

Proof : If $mP_m = 0$, the optimal portfolio strategy would be linear in τ as follow,

$$\hat{\pi}^0(\tau) = \frac{h_d - h_u}{u+d}\tau + \frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d}) \tag{31}$$

For the case $mP_m \neq 0$, the optimal portfolio strategy $\hat{\pi}$ can be divided into two parts, $\hat{\pi}^0$ and additional position θ . Then, given the first order condition,

$$0 = uP_u e^{-\gamma[(\hat{\pi}^0+\theta)u+\tau h_u]} + mP_m e^{-\gamma[(\hat{\pi}^0+\theta)m+\tau h_m]} - dP_d e^{-\gamma[-(\hat{\pi}^0+\theta)d+\tau h_d]}$$

we multiply $e^{\gamma[(\hat{\pi}^0+\theta)m+\tau h_m]}$ on both sides,

$$\begin{aligned}
0 &= uP_u e^{-\gamma[\hat{\pi}^0(u-m)+\theta(u-m)+\tau(h_u-h_m)]} - dP_d e^{\gamma[\hat{\pi}^0(m+d)+\theta(m+d)-\tau(h_d-h_m)]} + mP_m \\
&= uP_u e^{-\gamma[(\frac{h_d-h_u}{u+d}\tau+\frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d}))(u-m)+\theta(u-m)+\tau(h_u-h_m)]} - dP_d e^{\gamma[(\frac{h_d-h_u}{u+d}\tau+\frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d}))(m+d)+\theta(m+d)-\tau(h_d-h_m)]} + mP_m \\
&= uP_u(\frac{dP_d}{uP_u})^{\frac{u-m}{u+d}} e^{-\gamma[\tau\frac{u-m}{u+d}(h_d-h_u)+\tau(h_u-h_m)+\theta(u-m)]} - dP_d(\frac{uP_u}{dP_d})^{\frac{m+d}{u+d}} e^{\gamma[\tau\frac{d+m}{u+d}(h_d-h_u)-\tau(h_d-h_m)+\theta(m+d)]} + mP_m \\
&= (uP_u)^{\frac{m+d}{u+d}}(dP_d)^{\frac{u-m}{u+d}} e^{-\gamma\tau(\frac{m+d}{u+d}h_u+\frac{u-m}{u+d}h_d-h_m)}(e^{-\gamma\theta(u-m)} - e^{\gamma\theta(m+d)}) + mP_m
\end{aligned}$$

Therefore, θ must satisfies following equation,

$$e^{\gamma\theta(m+d)} - e^{-\gamma\theta(u-m)} = mP_m(uP_u)^{-\frac{m+d}{u+d}}(dP_d)^{\frac{m-u}{u+d}} e^{\gamma\tau(\frac{m+d}{u+d}h_u+\frac{u-m}{u+d}h_d-h_m)} \equiv \frac{mP_m}{\bar{K}}e^{\gamma\tau\bar{H}} \tag{32}$$

where $\bar{K} = (uP_u)^{\frac{m+d}{u+d}}(dP_d)^{-\frac{m-u}{u+d}}$ and $\bar{H} = \frac{m+d}{u+d}h_u + \frac{u-m}{u+d}h_d - h_m$.

If $m = \frac{u-d}{2}$, then $m+d = u-m$. By $\frac{1}{2}(e^x - e^{-x}) = \sinh(x)$, we have explicit θ,

$$\theta = \frac{2}{\gamma(u+d)}\sinh^{-1}(\frac{mP_m}{2\bar{K}}e^{\gamma\tau\bar{H}}) \tag{33}$$

□

To ensure the incompleteness of the market, condition ensuring the derivative not replicable is needed. If the derivative is replicable, there exists an trading strategy ζ satisfies

$$\begin{cases} h_u = \zeta(1+u) \\ h_m = \zeta(1+m) \\ h_d = \zeta(1-d) \end{cases} \Leftrightarrow \bar{H} = 0$$

which means the payoff of derivative in middle state is a linear combination of payoff in the other two states.

As mentioned in **3.3**, the key to generalize Gärtner-Ellis theorem is identifying converging rate r_n to set the indifference price $p^n(lr_n) \to p^\infty(l)$.

If there is a sequence $\{\tau_n := lr_n\}_{n=1,2,\dots}$, with $r_n \to \infty$ and associated probability $\{P_u^n, P_m^n, P_d^n\}$, then a corresponding sequence $\{\theta_n\}_{n=1,2,\dots}$ can be found via (29),

$$e^{\gamma\theta_n(m+d)} - e^{-\gamma\theta_n(u-m)} = \frac{m}{\bar{K}} e^{-\log P_m^n[\gamma\bar{H}l(\frac{r_n}{-\log P_m^n})-1]}$$

If r_n takes value $-\log P_m^n$ and τ_n is $-l\log P_m^n$, then above equation becomes

$$e^{\gamma\theta_n(m+d)} - e^{-\gamma\theta_n(u-m)} = \frac{m}{\bar{K}} e^{-\log P_m^n(\gamma\bar{H}l-1)} \tag{34}$$

As the market become asymptotically complete, i.e. $P_m \to 0$ in our trinomial case, $\frac{m}{\bar{K}} e^{-\log P_m^n(\gamma\bar{H}l-1)}$ blows up to $+\infty$, if $\gamma\bar{H}l > 1$. In this case, (31) only holds if $\theta_n = \frac{\gamma\bar{H}l-1}{\gamma(m+d)}(-\log P_m)$ or $\theta_n = \frac{\gamma\bar{H}l-1}{\gamma(u-m)}(\log P_m)$. If $\gamma\bar{H}l < 1$, $\frac{m}{\bar{K}} e^{-\log P_m^n(\gamma\bar{H}l-1)}$ converges to 0 and d equals $-u$, which violated our assumptions.

Now, the indifference price can be calculated in the trinomial case.

$$p(\tau) = -\frac{1}{\gamma\tau}\log(\frac{\mathcal{U}(0,\tau)}{\mathcal{U}(0,0)}) = -\frac{1}{\gamma\tau}\log(\frac{\mathbb{E}[e^{-\gamma\hat{X}_T^\tau-\gamma\tau h}]}{\mathbb{E}[e^{-\gamma\hat{X}_T^0}]}) \tag{35}$$

where

$$\begin{aligned}
\mathbb{E}[e^{-\gamma\hat{X}_T^\tau-\gamma\tau h}] &= P_u e^{-\gamma(\hat{\pi}u+\tau h_u)} + P_m e^{-\gamma(\hat{\pi}m+\tau h_m)} + P_d e^{-\gamma(-\hat{\pi}d+\tau h_d)} \\
\text{(By first order condition)} &= \frac{m-u}{m} P_u e^{-\gamma[(\hat{\pi}_0+\theta)u+\tau h_u]} + \frac{m+d}{m} P_d e^{\gamma[(\hat{\pi}_0+\theta)d-\tau h_d]} \\
&= \frac{m-u}{m} P_u e^{-\gamma[\frac{h_d-h_u}{u+d}\tau+\frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d})]u-\gamma\theta u-\gamma\tau h_u]} + \frac{m+d}{m} P_d e^{\gamma[\frac{h_d-h_u}{u+d}\tau+\frac{1}{\gamma(u+d)}\log(\frac{uP_u}{dP_d})]d+\gamma\theta d-\gamma\tau h_d]} \\
&= \frac{m-u}{mu}(uP_u)^{\frac{d}{u+d}}(dP_d)^{\frac{u}{u+d}} e^{-\gamma\tau\frac{dh_u+uh_d}{u+d}} e^{-\gamma\theta u} + \frac{m+d}{md}(uP_u)^{\frac{d}{u+d}}(dP_d)^{\frac{u}{u+d}} e^{-\gamma\tau\frac{dh_u+uh_d}{u+d}} e^{\gamma\theta d} \\
&= (uP_u)^{\frac{d}{u+d}}(dP_d)^{\frac{u}{u+d}} e^{-\gamma\tau\frac{dh_u+uh_d}{u+d}} (\frac{m-u}{mu}e^{-\gamma\theta u} + \frac{m+d}{md}e^{\gamma\theta d})
\end{aligned}$$

and $\mathbb{E}[e^{-\gamma\hat{X}_T^0}]$ is above equation taking $\tau = 0$.

If we set $\tau = lr_n$ with $r_n = -\log P_m^n$, then when $n \to \infty$,

$$p^\infty(l) = \frac{dh_u+uh_d}{u+d} - \frac{d}{d+m}(\bar{H} - \frac{1}{\gamma l}) \tag{36}$$

It can be further simplified,

$$
\begin{aligned}
p^{\infty}(l) &= \frac{dh_u + uh_d}{u+d} - \frac{d}{d+m}(\frac{m+d}{u+d}h_u + \frac{u-m}{u+d}h_d - h_m - \frac{1}{\gamma l}) \\
&= h_m \frac{d}{d+m} + h_d(\frac{u}{u+d} - \frac{d(u-m)}{(d+m)(u+d)}) + \frac{d}{d+m}\frac{1}{\gamma l} \\
&= \frac{dh_m + mh_d}{d+m} + \frac{d}{d+m}\frac{1}{\gamma l}
\end{aligned}
$$

With asymptotically complete market, our limit indifference price is still related to the middle states of the market.

5. Continuous Case

In continuous case, we set $S_0 = 1$ and $S_1 = X + 1$, where X and payoff of derivative h has joint normal distribution as follow,

$$
(X, h) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_X \sigma_h \rho \\ \sigma_X \sigma_h \rho & \sigma_h^2 \end{pmatrix}\right) \tag{37}
$$

X can be further decomposed as $X = \sigma_X Z^{(1)}$ where $Z^{(1)} \sim \mathcal{N}(0,1)$. h^n can be decomposed as $h = AZ^{(1)} + BZ^{(2)}$, where $Z^{(2)}$ is orthogonal to $Z^{(1)}$ with distribution $\mathcal{N}(0,1)$ and $\sigma_h^2 = A^2 + B^2$. Then A and B can be specified as follow,

$$
A = \sigma_h \rho; \quad B = \sqrt{\sigma_h^2(1-\rho^2)} \tag{38}
$$

If we replace ρ with a sequence of $\{\rho_n\}$ converging to 1, the market becomes asymptotically complete, and the corresponding return and payoff are marked as (X^n, h^n).

In this case, our original optimal problem becomes

$$
\sup_{\pi} \mathbb{E}[-e^{-\gamma\pi X - \gamma\tau h}] \propto \mathbb{E}[e^{-\gamma\pi Z^{(2)}}] \cdot \inf_{\pi} \mathbb{E}[e^{-\gamma(\pi + \tau\frac{A}{\sigma_X})X}] \tag{39}
$$

From (36), it is easy to observe that $\hat{\pi}(\tau) = \hat{\pi}(0) - \tau\frac{A}{\sigma_X}$. Let's focus on the $\hat{\pi}(0)$, which solves

$$
\begin{aligned}
\inf_{\pi} \mathbb{E}[e^{-\gamma\pi X}] &= \inf_{\pi} \int e^{-\gamma\pi X} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{X^2}{2\sigma_X^2}} dX \\
&= \inf_{\pi} e^{\frac{\gamma^2\pi^2\sigma_X^2}{2}} \cdot \int \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(X+\gamma\pi\sigma_X^2)^2}{2\sigma_X^2}} dX \\
&= \inf_{\pi} e^{\frac{\gamma^2\pi^2\sigma_X^2}{2}}
\end{aligned}
$$

It can be easily observed that $\hat{\pi}(0) = 0$. Then $\hat{\pi}(\tau) = \hat{\pi}(0) - \tau\frac{A}{\sigma_X} = -\tau\frac{\sigma_h\rho}{\sigma_X}$, which is a linear function of τ. With optimal trading strategy $\hat{\pi}(\tau)$,

$$Y = \frac{1}{\tau}(\hat{X}_1^\tau - \hat{X}_1^0) + h = \frac{1}{\tau}(\hat{\pi}(\tau) - \hat{\pi}(0))X + h = h - \frac{\sigma_h\rho}{\sigma_X} \tag{40}$$

which is no longer a function of τ . So we need other more delicate distribution to generalize Gärtner-Ellis theorem in continuous setting.

6. Further Research

For the further research, one can be done is generalizing one-period trinomial model into multi-period trinomial model to see whether the Y is a function of τ. Furthermore, one can consider other distributions like double exponential distribution and mixed Gaussian distribution. As we can see from normal distribution case, market becomes asymptotically complete as correlation ρ converges to 1. In the same spirit, the key point for more general model is the condition that leads to asymptotically complete market.

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1. This report is a summary of a research project supervised by Professor Scott Robertson in 2019 summer.↩

2. PhD student in Questrom School of Business↩